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Symmetric Binary Forms and Involutions.—*Continued.**

BY ARTHUR B. COBLE.

§ 10. *The Form $H_{3,2}$ as a Ternary Quartic.*

In the representation of § 2, the symmetric form

$$H_{3,2} \equiv (a_1x_1)^2(a_2x_2)^2(a_3x_3)^2(a_4x_4)^2$$

is viewed as a ternary quartic with reference to an arbitrarily chosen proper norm-conic N. If no conditions are imposed on the form the quartic is general. An involutive set of the form determines a set of five points on N such that any four of the five are apolar to the quartic, *i. e.*, an antorthic five-point or $A^5_{4,4}$ of the quartic. The $A^5_{4,4}$'s are subject to five conditions and are ∞^5 in number. A conic will pass through each one. If N be chosen as such a conic, $H_{3,2}$ has one involutive set and therefore, according to (26), has ∞^1 involutive sets. Hence

(83) *The ∞^5 antorthic sets $A^5_{4,4}$ of a general quartic are distributed ∞^1 at a time on each of a system of ∞^4 conics. On each conic of the system the $A^5_{4,4}$'s constitute the involution $\overline{H}_{1,4}$ of* (26).

In the representation of § 3, the translation of (26) is:

(84) *If a pentahedron in S_4 has its five points on a quadric and its five S_3's on a norm-curve N_4, there are ∞^1 pentahedra with points on the quadric and S_3's on N_4.*

The next and last stage in the number of involutive sets of $H_{3,2}$ is when it has ∞^3 sets and becomes an $I_{3,2}$. Hereafter we shall suppose this to be the case. The quartic, Q, has then ∞^3 $A^5_{4,4}$'s on the norm-conic N. If a point x_4 on N be fixed, the $I_{3,2}$ reduces to an $I_{2,2}$ and N is an involution conic of a cubic, f, the polar cubic of x_4 as to N. Since the invariant, $\sqrt{\tfrac{1}{6}S}$, of this polar cubic is of the second degree in the coefficients of the $I_{2,2}$, there are four points on N whose polar cubics have $S=0$. These four points are determined by the equation

$$(85) \qquad (a_1a_1')(a_2a_2')[(a_1a_2)(a_1'a_2') + (a_1a_2')(a_1'a_2)](a_3a_3')^2(a_4x)^2(a_4'x)^2 = 0.$$

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* This article is a continuation of one under the same title in this volume, pp. 182–212. The numbering of paragraphs and theorems is consecutive with that of the previous paper.

Let the four parameters of points on N, determined by (85), be y_1, y_2, y_3, y_4. Since N is an involution conic of the polar cubic of y_4, for which $S = 0$, it must pass through the vertices of the Hessian triangle of the polar cubic. But these also are known to be points on the covariant S of the quartic Q and therefore must be the points y_1, y_2, y_3. Hence the four points are a symmetrical set on N; the polar conic of any two is the square of the line joining the other two or any three of the four are apolar to the quartic. *If $H_{3,2}$ is an $I_{3,2}$, the quartic Q has an antorthic set $A^4_{3,4}$.*[*] To show that this condition is sufficient and to find the number of conics N when it is satisfied, we employ the convenient canonical form of a quartic with an $A^4_{3,4}$, namely, a sum of the fourth powers of the six lines joining the four points. Let then the four points be the vertices of the triangle of reference and the unit point.

$$(86) \qquad Q = ax_1^4 + bx_2^4 + cx_3^4 + l(x_2 - x_3)^4 + m(x_3 - x_1)^4 + n(x_1 - x_2)^4.$$

Since the conic N must contain ∞^3 $A^5_{4,4}$'s, the polar conic of any two points y and z on N as to Q must be (when taken in line form) apolar to N. Since also N must pass through the $A^4_{3,4}$ its equation is

$$\alpha x_1(x_2 - x_3) + \beta x_2(x_3 - x_1) + \gamma x_3(x_1 - x_2),$$

where

$$\alpha + \beta + \gamma = 0.$$

Then if y and z are two points on N, we have

$$(\beta - \gamma) : (\gamma - \alpha) : (\alpha - \beta) =$$
$$y_1(y_2 - y_3) : y_2(y_3 - y_1) : y_3(y_1 - y_2) = z_1(z_2 - z_3) : z_2(z_3 - z_1) : z_3(z_1 - z_2).$$

Forming the line equation of the polar conic of y and z as to Q and substituting for the coordinates of y and z their values in terms of α, β, γ, the apolarity condition reduces to

$$(87) \qquad al(\beta - \gamma)^3 + bm(\gamma - \alpha)^3 + cn(\alpha - \beta)^3 = 0.$$

Hence *if Q has an $A^4_{3,4}$, there are three conics on the $A^4_{3,4}$ each of which contains ∞^3 $A^5_{4,4}$'s of Q which constitute an $I_{3,2}$.* Such conics will be called *involution conics of Q.*

The binary cubic (87) is apolar to the cubic $(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)$ and its Hessian is

$$bcmn(\gamma - \alpha)(\alpha - \beta) + canl(\alpha - \beta)(\beta - \gamma) + ablm(\beta - \gamma)(\gamma - \alpha).$$

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[*] Quartics of this type have been noticed briefly by Caporali; *Memoire*, p. 347, §§35, 36, 37, 40 and 44.

But $\beta - \gamma = 0$ determines the parameter of a degenerate conic in the pencil through $A^4_{3,4}$. Hence the two conics of the pencil which form the Hessian pair of the three involution conics are

$$bcmn\, x_2 x_3 (x_3 - x_1)(x_1 - x_2) + \ldots,$$

which according to Caporali is the covariant S. Thus the triad of involution conics is completely defined by its apolarity with the triad of line pairs on $A^4_{3,4}$ and with the pair of conics into which S degenerates.

That Q have an $A^4_{3,4}$ is a single invariant condition which is easily deduced from the canonical form (86). Denote this invariant by S_1 because of its resemblance to the invariant S of a cubic. Caporali calculates the invariant of Q of the third degree

$$A = C_{3,0,0} = (a + b + c)(mn + nl + lm) + al(b + c) + bm(c + a) + cn(a + b) + abc.$$

The invariant B of degree six which is the condition that Q have an apolar conic is

$$B = C_{6,0,0} = abclmn.$$

The covariant conic of degree five which, when $B = 0$, is the point equation of the apolar conic is

$$C = C_{5,2,0} = B \left\{ \frac{x_1^2}{l} + \frac{x_2^2}{m} + \frac{x_3^2}{n} + \frac{(x_2 - x_3)^2}{a} + \frac{(x_3 - x_1)^2}{b} + \frac{(x_1 - x_2)^2}{c} \right\}.$$

The invariant D of degree fifteen, the discriminant of C, turns out to be B^2A, *i.e.*,

$$D - B^2 A^{*} = 0.$$

Since in general D does not factor into B^2A,[†] this is an invariant condition satisfied by our special quartic and furnishes the required invariant, $S_1 = D - B^2A$, of degree fifteen. Summing up the above we find that

(88) *The involution form, $I_{3,2}$, with reference to a norm-conic, N, represents a ternary quartic which satisfies the single invariant condition of degree fifteen, $S_1 = 0$. Q has an $A^4_{3,4}$ and the covariant S of Q is two conics on the $A^4_{3,4}$. In the pencil of conics on $A^4_{3,4}$ there is a set of three which is apolar to the set of three degenerate conics of the pencil and whose Hessian pair is S. On each of these three involution conics, one of which is N, the sets $A^5_{4,5}$ of Q lie in an $I_{3,2}$.*

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[*] The numerical factors of the invariants are not adjusted to any given system.

[†] *E. g.*, D is irreducible for the special quartic of Salmon which contains only even powers.

The simplest picture of an $I_{3,2}$ is the system of intersections of a rational quintic R_5 in S_3 by the planes of S_3. Hence we have shown that

(89) *The rational invariant theory of the rational quintic in S_3 coincides with that of a ternary quartic subject to the condition $S_1 = 0$ after the adjunction of the cubic irrationality which separates the three involution conics.**

To every comitant of Q there corresponds a binary comitant of the form $I_{3,2}$, the transition being made with formulae (29) and (30). And to every comitant of $I_{3,2}$ there corresponds a comitant of the ternary form Q which may, however, according to (89), be irrational. Each covariant of Q cuts the involution conic N in a set of points whose parameters are the parameters of a set of covariant points on R_5; *e.g.*, Q itself determines on N the parameters of the eight hyperosculating planes on R_5. The Hessian of the polar cubic of any point x_1 on N cuts N in three pairs of corresponding points; each pair with x_1 determines the parameters of a triple secant of R_5. The covariant S determines on N the parameters of the four meets with R_5 of its unique fourfold secant.

Let us consider the 24 meets of N and the Steinerian of Q, $R = T^2 - \frac{1}{6}S^3$, a curve of order twelve. If x_1 is the parameter on N of one of the 24 points, for this fixed x_1, $I_{3,2}$ becomes an $I_{2,2}$ represented by the nodal polar cubic of x_1 for which N is an involution conic. Since the discriminant of f (for any point x_1 on N) breaks up† into two rational factors of degree six, the 24 points separate into two sets of 12 points. The involution conic passes through the nodes of the polar cubics of the points of the one set. The node and flex-line of the polar cubics of points of the other set are pole and polar with regard to N [see pp. 205–6]. Hence

(90) *Each of the three involution conics of a ternary quartic with an $A^4_{3,4}$ meets the Steinerian in the* 12 *points* (β) *which correspond to the* 12 *points* (α) *in which it meets the Hessian. It meets the Steinerian further in* 12 *points* (γ) *of whose nodal polar cubics the node and flex-line are pole and polar as to the conic.*

To the sets (α), (β), (γ) on N there correspond on R_5 sets of twelve points described as follows:

(91) *R_5 has twelve tangents at points* (α) *which meet the curve again at points* (β).

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*In some special cases noted hereafter the part of this theorem which concerns the number of involution conics requires modification.

†This separation does not involve a degeneration of R into two curves of order 6. Similarly for any point x_1 on N the invariant S of the polar cubic of x_1 is the square of a rational expression, but the covariant S is not the square of a conic.

(92) *A triple secant of* R_5 *on a point* r *meets* R_5 *in points* s_1 *and* s_2 *and contains two planes which touch* R_5 *at points* t_1 *and* t_2*; there are* 12 *points* r*, points* (γ)*, such that the three triple secants on* r *determine three pairs* s_1s_2 *involutorily projective to the three pairs* t_1t_2*.*

To every invariant of Q there corresponds an invariant of R_5. Let us examine for example the invariant B. When $S_1 = 0$ and Q is in the canonical form (86), clearly $B = 0$ requires that one of the six coefficients, say l, be zero. $B = 0$ is also the condition that Q have an apolar conic which in this case is the point pair u_2u_3, two points of $A^4_{3,4}$. The conic C is then x_1^2 and it cuts N in the apolar point pair whose parameters are, say x_1 and x_2. The polar line of the three points x_1, t_1, t_2, where t_1 and t_2 are arbitrarily chosen on N, will cut N in the points x_2 and t_3, *i. e.*, every one of the set of $A^5_{4,5}$'s which contain x_1 contains also x_2. This requires on R_5 that every plane on the point whose parameter is x_1 meet the curve again in the point whose parameter is x_2, whence these are the parameters of a double point on R_5.

(93) *The condition that* R_5 *have a double point is the vanishing of the invariant* B *of the sixth degree in the coefficients of its* $I_{3,2}$*. The parameters of the double point are determined from the two double roots of* $(c_1x)^2(c_2x)^2 = 0$*, where* $(c_1x_1)^2(c_2x_2)^2$ *is the symmetric form of degree five in the coefficients of* $I_{3,2}$ *which represents the conic* C*.*

Instead of using known properties of Q to obtain facts concerning R_5 the reverse process can be employed. It is well known that an R_5 can lie on a quadric surface, in which case all generators of one kind are fourfold secants. The $I_{3,2}$ of such an R_5 determines an interesting ternary quartic not noticed hitherto. As before, to each fourfold secant, there corresponds on N four double points of S, whence S must be the square of the conic N and its points must fall into sets of four which belong to an $I_{1,3}$. Each set of the $I_{1,3}$ is an $A^4_{3,4}$ of Q.

We shall show first that S the square of a conic is a sufficient condition for this phenomenon, the necessity of the condition being obvious. Let s_1 be any point on the conic S. The polar cubic of s_1 as to Q has a Hessian triangle whose vertices, t_1, t_2, t_3, also lie on S. Each set of three points, $s_1t_it_k$, is apolar to Q. The polar cubic of t_1 has a Hessian triangle with vertices $s_1\tau_2\tau_3$ on S. But s_1t_2 and s_1t_3 are apolar to this cubic, whence $\overline{t_2t_3}$ is the line of the Hessian triangle opposite s_1, *i. e.*, the lines $\overline{t_2t_3}$ and $\overline{\tau_2\tau_3}$ coincide. Hence the pairs t_2t_3 and $\tau_2\tau_3$ on the same line and conic coincide. Thus any three of the four points $s_1t_1t_2t_3$ are the Hessian triangle of the polar cubic of the remaining point and the four are an $A^4_{3,4}$ of Q. Evidently there are ∞^1 such $A^4_{3,4}$'s of Q on S in an $I_{1,3}$. That S is an involution conic follows from the fact that S is on the Hessian triangle of the polar cubic of any point on S and therefore is an involution conic of the polar cubic (p. 204). But if an $H_{3,2}$ for any assigned value of one variable reduces to an $I_{2,2}$, the $H_{3,2}$ is an $I_{3,2}$. Hence

(94) *That the covariant S of a quartic Q be the square of a conic is the necessary and sufficient condition that Q have $\infty^1 A^4_{3,4}$'s which must lie in an $I_{1,3}$ on the conic. The conic also contains $\infty^3 A^5_{4,5}$'s of Q which lie in the $I_{3,2}$ of an R_5 on a quadric surface.*

Further it is clear that

(95) *An $I_{1,3}$ on a conic determines a unique quartic $\bar{Q}$ whose covariant S is the square of the conic. The sets of $I_{1,3}$ are $A^4_{3,4}$'s of $\bar{Q}$.*

The R_5 on a quadric, the $I_{1,3}$, and the quartic $\bar{Q}$ each depend on three absolute constants. The quartic Q with a unique $A^4_{3,4}$ depends on five absolute constants. It is one condition on a, b, c, l, m, n that S be the square of a conic. Writing

$$\begin{aligned} r_1 &= al, & r_2 &= bm, & r_3 &= cn, \\ s_1 &= r_1 + r_2 + r_3, & s_2 &= r_2r_3 + r_3r_1 + r_1r_2, & s_3 &= r_1r_2r_2, \end{aligned}$$

this condition is $s_2^2 - 4s_1s_3 = 0$. When this is satisfied, the number of constants is four; but the reduction to the canonical form is possible in ∞^1 ways, whence the number of absolute constants is three. In treating $\bar{Q}$ we must suppose $B \neq 0$, which emphasizes that an R_5 on a proper quadric can not have a double point.

The Steinerian and Hessian of $\bar{Q}$ have remarkable properties. It will happen six times that two points of an $A^4_{3,4}$ coincide. Let the coincident pair of such an $A^4_{3,4}$ be ρ_i and the other two points be σ_i and τ_i, $i = 1, 2, \ldots, 6$. Then $\rho_i^2\sigma_i$, $\rho_i^2\tau_i$, and $\rho_i\sigma_i\tau_i$ are each apolar to $\bar{Q}$. This requires that the polar cubics of σ_i and τ_i have cusps at ρ_i with cusp tangents $\overline{\rho_i\tau_i}$ and $\overline{\rho_i\sigma_i}$ respectively. So all points on the line $\overline{\sigma_i\tau_i}$ have polar cubics with a node at ρ_i, and the line is a part of the Steinerian. Since the Jacobian of a net of curves has a double point at a point where all the curves of a pencil in the net have a double point, the Hessian of $\bar{Q}$ (the Jacobian of the net of first polars) has a double point at ρ_i. Evidently the Steinerian, $R = T^2 - \frac{1}{6}S^3$, factors into two sextics when S is the square of a conic. The factors are $R_1 = T + \sqrt{\frac{1}{6}S}\,S$ and $R_2 = T - \sqrt{\frac{1}{6}S}\,S$. We shall see that, to accord with our earlier convention,[*] the first factor must represent the six lines $\overline{\sigma_i\tau_i}$. Since $T=0$ and $R_2=0$, when $R_1=S=0$, we find that the line $\overline{\sigma_i\tau_i}$ must meet both R_2 and T three times at σ_i and three times at τ_i. Thus R_2 and T each have the line $\overline{\sigma_i\tau_i}$ for a double flex-tangent at the flex-points σ_i and τ_i. Since the cusp tangents of first polars touch the Hessian at the cusp, the tangents to the Hessian at ρ_i are $\overline{\rho_i\tau_i}$ and $\overline{\rho_i\sigma_i}$.

(96) *The Hessian of Q has six double points ρ_i on the conic S whose tangents cut S again in pairs of points $\sigma_i\tau_i$. The Steinerian decomposes into the six lines $\overline{\sigma_i\tau_i}$ and a sextic R_2 which has the six lines $\overline{\sigma_i\tau_i}$ for double flex-tangents which touch at σ_i and τ_i. R_2, being in one-to-one correspondence with H, has six further double points not on S. The polar cubics of points on $\overline{\sigma_i\tau_i}$ have double points at ρ_i, the double-point tangents being harmonic with $\overline{\rho_i\tau_i}$ and $\overline{\rho_i\sigma_i}$, which are the cusp tangents of the polar cubics of σ_i and τ_i respectively.*

According to (94), $N=\sqrt{S}$ is an involution conic of $\bar{Q}$. But $\bar{Q}$ has also other involution conics. From the definition of the involution conics in (88) we conclude that when S is a perfect square two of the three involution conics on an $A^4_{3,4}$ coincide with S. But the third is distinct and still has the property that taken with the conic S twice it forms a triad apolar to the three degenerate conics on the $A^4_{3,4}$. But this requires that the involution conic be apolar to S in line form.[†] The four points on S being given, there is a single conic of this kind and the $I_{1,3}$ of $A^4_{3,4}$'s determines a pencil of conics M which meet in a set of four points, O^4, orthic[‡] to the conic S. On an involution conic M there is a single $A^4_{3,4}$ of $\bar{Q}$ and its involution is that of an R_5 which does not lie on a quadric. As before, we show that the twelve meets of M and H correspond to the twelve meets of M and R_2, whence the sextics H and R_2 are cut by ∞^1 conics in twelve pairs of corresponding points.

Through a point x on a line $\overline{\sigma_i\tau_i}$ there passes a conic M which is an involution conic of the polar cubic of x. The invariant R_1 of this cubic vanishes;

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[*] In the first paper the general cubic was considered with regard to a *proper* involution conic, the polar conic of a point P as to the cubic A. If, however, P were on the Hessian of A, Δ_A, its polar conic would be degenerate and the entire construction would fail. From (54), $\Delta_A=-\frac{1}{3}R_1(B-2\sqrt{\frac{1}{6}Sf})$. The condition that P lie on Δ_A is given in (52). Comparing this condition with (49) we find it to be $R_1=0$. For this case, however, a new determination of the involution conics was made (p. 205).

[†] A triad of conics in a pencil, $(ax)^2$, $(bx)^2$, $(cx)^2$, is apolar to the triad of degenerate conics in the pencil if $(abc)^2=0$.

[‡] *I. e.*, S in lines is a sum of squares of the four points.

hence its flex-line is the polar of its node ρ_i as to M. The polar lines of ρ_i as to the pencil M meet in a point r_i, the partner of ρ_i in the quadratic Cremona involution determined by O^4. The line $\overline{r_i\rho_i}$ is touched by two conics of the pencil at r_i and ρ_i. Since the conic on ρ_i touches N at ρ_i, the line $\overline{r_i\rho_i}$ is the tangent to $N = \sqrt{S}$ at ρ_i. This is also evident from the fact that this tangent is the flex-line of the cuspidal polar cubics of σ_i and τ_i.

(97) *In addition to the special involution conic N whose square is S, Q has ∞^1 involution conics M which lie in a pencil apolar to N in lines and cut N in the $A^4_{3,4}$'s of $\bar{Q}$. The conics M each cut the Steinerian sextic R_2 and the Hessian in twelve pairs of corresponding points.*

The flex-lines of the polar cubics of points on a line $\overline{\sigma_i\tau_i}$ meet in a point r_i, each line on r_i being a flex-line of two cubics. r_i is on the tangent to N at ρ_i and the pair of points $r_i\rho_i$ is apolar to all the conics M.

We shall show further that the six Steinerian lines touch a conic. Take ρ_i, σ_i, τ_i as the reference triangle u_1, u_2, u_3 and let $x_2 + x_3 = 0$ be the tangent to N at u_1. Then $\bar{Q}$ takes the form

$$\alpha x_1^4 + \beta x_2^4 + \gamma x_3^4 + \delta(x_2 + x_3)^4 + 4\lambda x_2^3 x_1 + 4\mu x_3^3 x_1.$$

$$B = \alpha\delta\lambda^2\mu^2$$

and

$$C = \alpha\lambda^2\mu^2 x_1^2 - \alpha\delta\gamma\lambda^2 x_2^2 - \alpha\delta\beta\mu^2 x_3^2 + 2\alpha\delta\lambda\mu\, x_1(\lambda x_2 + \mu x_3).$$

Thus $x_1 = 0$ meets C in two points apolar to u_2u_3, the two points in which it meets N. Hence this line lies on the conic locus of lines which cut N and C in harmonic pairs.

(98) *The six Steinerian lines touch a conic, the intermediate or Clebschian of C and N.*

Further study of the comitants of $\bar{Q}$ would no doubt yield results worth noting. For example the point equation of the contravariant ϕ of degree 5 and class 4 has been determined[*] to be

$$2(3BS - 2C^2)(2HC - S^2) - (3BH - SC)^2 = 0.$$

Its twenty-four cusps are the meets of $3BS - 2C^2 = 0$ and $3BH - SC = 0$. From the equation of ϕ we find that it has the six double points of H as double points with the same tangents as H. Also since $3BS - 2C^2$ factors, the 24 *cusps of ϕ lie* 12 *at a time on two conics of the pencil determined by $\sqrt{S}$ and C.* The locus of lines whose associate conics degenerate is in general a sextic ψ. The ψ of $\bar{Q}$ is the square of the cubic whose lines join pairs of points in the $I_{1,3}$ on N.

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[*] Coble, AMERICAN JOURNAL, Vol. XXVIII, pp. 344–45.

The above discussion suggests a classification of non-singular R_5's into three types:

(99) *Type* (*A*). R_5 is the general quintic with a single fourfold secant. Q is the general quartic with an $A^4_{3,4}$ and three involution conics (90).

Type (*B*). R_5 has a single fourfold secant. Q has ∞^1 $A^4_{3,4}$'s. The involution conic is one of a pencil and contains a single $A^4_{3,4}$. R_5 depends on four constants, Q depends on three constants and the involution conic on one.

Type (*C*). R_5 has ∞^1 fourfold secants on a quadric. The involution conic is unique. R_5 and Q each depend on three constants.

In the usual classification according to the number of fourfold secants, types (*A*) and (*B*) coincide. In this classification with reference to Q, types (*B*) and (*C*) hang together.

To characterize the R_5 of type (*B*) we observe that on the involution conic M the 12 Steinerian points whose corresponding Hessian points do not lie on M are cut out by the six Steinerian lines and separate into six pairs. The polar cubics of a pair have the same flex-line. But this line cuts M in the fixed points of the binary involution described in (73) and (92). Hence

(100) *On an R_5 of type (B) the 12 points (γ) of (92) separate into six pairs, the points of a pair determining the same binary involution on the curve.*

This is not a very marked geometrical peculiarity of the curve. A better condition should arise from the existence of the four special points of the curve which correspond to the base points of the pencil of involution conics.

When R_5 has an actual multiple point, the number of $A^4_{3,4}$'s of Q and the number of involution conics vary.

If R_5 has a double point, Q has an apolar point pair, $p_1 p_2$. The polar cubic of either point is three lines through the other. The Hessian pairs of the two sets of three lines form a quadrilateral with opposite vertices p_1, p_2; q_1, q_2; and r_1, r_2. Then both p_1, p_2, q_1, q_2 and p_1, p_2, r_1, r_2 form an $A^4_{3,4}$ of Q. On each $A^4_{3,4}$ there are three involution conics. S is the product of the four sides of the quadrilateral.

If R_5 has two double points, Q is a sum of the fourth powers of four lines. Any two of the three pairs of opposite vertices of the quadrilateral are an $A^4_{3,4}$ of Q, there being three involution conics on each $A^4_{3,4}$. Again S is the product of the four lines.

If R_5 lies on a quadric cone and has a triple point at the vertex, Q is a sum of three fourth powers. S vanishes identically. Any conic through the vertices of the three-line is an involution conic. The vertices determine the parameters of the triple point.

I will obtain finally a covariant conic of Q which becomes S when S is the square of a conic. Taking Q in the canonical form (86) and using the notation of p. 360 we find that the conic whose polar as to Q is C is

$$K = B^2 \left\{ \Sigma \frac{1}{r_1} [u_1(u_1 + u_2 + u_3) - u_2 u_3] \right\}.$$

This conic and S involve the six coefficients only in the combinations r_1, r_2, r_3. The point equation of K is

$$\begin{aligned} C_1 \equiv B^4 \Big\{ &\Sigma \frac{1}{r_1^2} [-3x_1^2 + x_2^2 + x_3^2 - 3(x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2] \\ &+ \Sigma \frac{2}{r_2 r_3} [-x_1^2 + x_2^2 + x_3^2 - (x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2] \Big\}. \end{aligned}$$

The polar of K as to S is

$$\begin{aligned} C_2 \equiv B^3 \Big\{ &\Sigma \frac{1}{r_1^2} [-3x_1^2 + 2x_2^2 + 2x_3^2 - 3(x_2 - x_3)^2 + 2(x_3 - x_1)^2 + 2(x_1 - x_2)^2] \\ &+ \Sigma \frac{1}{r_2 r_3} [-4x_1^2 + x_2^2 + x_3^2 - 4(x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2] \Big\}. \end{aligned}$$

Then

$$C_1 + BC_2 = 6B^2 \{ \Sigma (r_1^2 r_2^2 - r_1^2 r_3^2 - r_1 r_2^2 r_3 + r_1 r_2 r_3^2) x_1 (x_2 - x_3) \}.$$

Evidently this covariant conic of degree twenty goes through $A^4_{3,4}$. When Q has ∞^1 $A^4_{3,4}$'s, it must either vanish identically or pass through all and coincide with S. But one easily verifies that its vanishing imposes further conditions on r_1, r_2, r_3 than the one, $s_2^2 - 4s_1 s_3 = 0$, which requires that S be the square of a conic. Hence it must coincide with S.

A discussion of the relation between a ternary rational quintic and the cubic surface determined by its $I_{2,3}$ is reserved for a later article.

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